The Fellow. Yes, something from the neutron-star merger should be rare and I wouldn't expect the whole galaxy to be enriched in the same way by neutron-star mergers.

Dr. Ness. I think there are two points. Yes, all of the neutron-capture elements have higher scatter, but we are predicting on non-neutron-capture elements. I think in the residuals, the fact they're all so correlated, implies that they're all generated from the same underlying 'other source'. I think that we can do that test, but to do that we need really to look at a range of metallicities and look at how many stars are failing our prediction — about 1% fail. In particular, we see things fail in barium and yttrium because that's produced, I think, in stellar-companion mass transfer. I think that we can test that. We just haven't yet.

The President. We should thank Melissa again for a fabulous talk [applause]. I have some questions for you, but I will ask you later today. The next meeting will be 11th April in the Royal Irish Academy, Dublin.

A SIMPLE, STABLE, RAPIDLY-CONVERGING, AND EXTREMELY ACCURATE ITERATIVE SOLUTION FOR KEPLER'S EQUATION

By B. Cameron Reed

Department of Physics, Alma College, Michigan

A very straightforward scheme for iteratively solving Kepler's equation is described. The iteration method is the familiar Newton-Raphson technique, augmented with an initial solution estimate based on modelling the sine function as a downward-opening parabola. For even extreme eccentricities, micro-arcsecond accuracy can be achieved with about a dozen iterations. No use is made of 'canned' calculator or spreadsheet root-finding algorithms.

It is well known that it is impossible to obtain a closed-form solution of Kepler's equation for the eccentric-anomaly position angle E of an orbiter as a function of time t. This problem has accumulated four centuries of analysis, and a student or non-specialist researcher who explores the on-line, textbook, and journal literature on it will soon come across intimidating-looking series expansions with coefficients involving Bessel functions, dire warnings concerning convergence instabilities, and a plethora of strategies for generating an initial estimate for E.[1,2,3]

Solving Kepler's equation is now trivial given the root-finding routines available in calculators and spreadsheets. As an instructional strategy, however, defaulting to 'black box' solutions whose inner workings are opaque has obvious disadvantages. The purpose of this note is to describe a very simple, rapidly-converging, iterative solution to Kepler's equation that makes use of an intuitively-appealing procedure for generating an initial guess. The procedure converges to an accuracy of a few micro-arcseconds in about a dozen iterations even in cases of extreme eccentricity at times close to periapsis, when E is evolving rapidly. The iterative engine is the familiar Newton-Raphson method; what may be new (or, more likely, not as well-appreciated as it should be) is the initial-guess procedure. My intent here is not to supersede 400 years of analysis of this problem, but rather to offer a scheme that can be described and implemented in a spreadsheet without appeal to any built-in root-finding routines within an ordinary lecture period.

For an orbit of eccentricity ε and period T, Kepler's equation is

$$f(E) = E + \varepsilon \sin E - 2\pi\tau, \tag{1}$$

where $\tau = t/T$. In this formulation, the force centre is at the left focus of the ellipse; apapsis corresponds to $E = 0$ at $t = \tau = 0$, and periapsis to $E = \pi$ at $\tau = 1/2$. From the symmetry of an ellipse, we need concern ourselves only with the range $0 \le \tau \le 1/2$ and $0 \le E \le \pi$. For times in the range $1/2 < \tau \le 1$, simply follow the procedure outlined in what follows but use $1-\tau$ in place of the desired value of τ, and then negate the result for E.

As for generating an initial guess for E, I appeal to the fact that sin E for $0 \le E \le \pi$ is very similar to a downward-opening parabola. This is illustrated in Fig. 1, where the solid curve shows sin E and the dashed curve a parabola fit to pass through the end points sin $E = 0$ at $E = 0$ and π, while having a value of

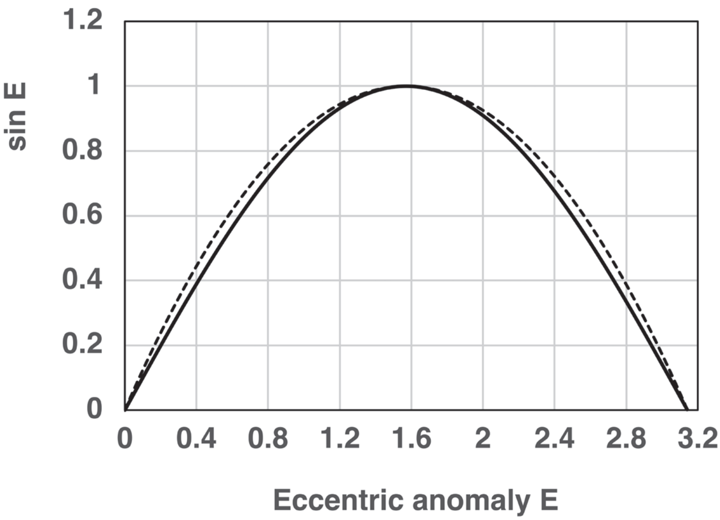

FIG. 1

sin E *vs.* E (solid curve) and the approximation of Eq. (2) (dashed curve).

unity at $E = \pi/2$. The resulting approximation is

$$\sin E \approx -\frac{4}{\pi^2} E^2 + \frac{4}{\pi} E. \tag{2}$$

On substituting this into Eq. (1), the result is a quadratic equation in E that can be solved for an initial guess E_0, the 'zeroth' iteration. The form is $AE^2 + BE + C = 0$ with coefficients

$$[A, B, C] = [\frac{4\varepsilon}{\pi^2}, -(1 + \frac{4\varepsilon}{\pi}), 2\pi\tau]. \tag{3}$$

Once E_0 has been determined, subsequent iterations proceed by the usual Newton method,

$$E_{n+1} = E_n - \frac{f(E_n)}{f'(E_n)}, \tag{4}$$

where the prime denotes the derivative of Eq. (1) with respect to E, $1 + \varepsilon \cos E$.

Selecting the positive sign in the numerator of the solution of the quadratic for E_0 always gives a value of E greater than π, which can be discarded given the operative range of $0 \leq E \leq \pi$. Although this root does converge to π for $\varepsilon = 1$ and $\tau = ½$ and so might be the root of choice for values of (ε, τ) near those limits, it has the danger of flirting with the divergence of the derivative in Eq. (4) at periapsis. For this reason, I worked exclusively with the negative-sign root, which converges to $\pi^2/4 \sim 2{\cdot}467$ for $\varepsilon = 1$ and $\tau = ½$.

I constructed a spreadsheet to implement this scheme. Experimentation shows that for non-extreme values of ε and τ, micro-arcsecond convergence is reached in two or three iterations. Fig. 2 shows results from running an extreme example, $(\varepsilon, \tau) = (0{\cdot}999999, 0{\cdot}499999)$. The plot shows the logarithm of the change in the number of arcseconds in the estimate of E from the input

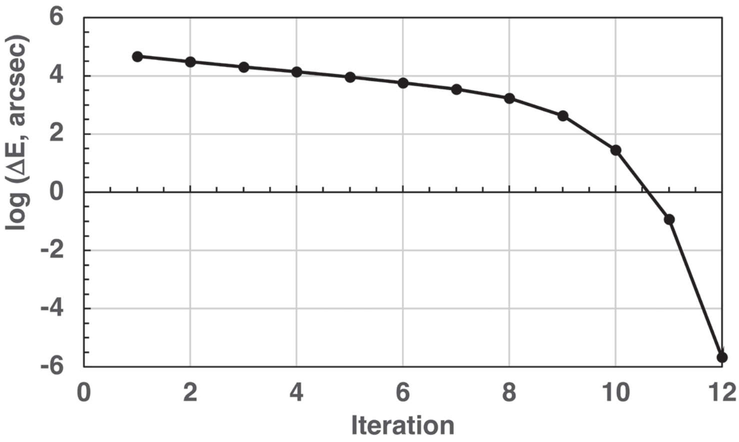

Fig. 2

Logarithm of iteration-to-iteration change in estimate of E in seconds of arc.

value at a given iteration to the succeeding one. The initial estimate is E_0 = 141°·370493, which iterates to E_1 = 154°·443789, for a change of 13°·073296 = 47063·9 arcseconds and a change in the logarithm of 4·673; this is plotted at the location of iteration 1. Arcsecond-level convergence is achieved on going from the 10th to the 11th iteration, and micro-arcsecond convergence on going from the 11th to the 12th; at this point the procedure is running up against the limitation of the computer's accuracy of about 1 part in 10^{15}. At convergence, E = 178°·082209… . The author would be happy to share the spreadsheet with any interested reader.

In conclusion, solving Kepler's equation need not be daunting or mysterious. Sometimes, a simple approach to an old problem can yield perfectly respectable results.

References

(1) https://en.wikipedia.org/wiki/Kepler%27s_equation
(2) See, for example, A. W. Odell and R. H. Gooding, 'Procedures for Solving Kepler's Equation' in *Cel. Mech.*, **38**(4), 307, 1986.
(3) E. D. Charles & J. B. Tatum, 'The Convergence of Raphson Iteration with Kepler's Equation' in *Cel. Mech.*, **69**(4), 357, 1998.

REDISCUSSION OF ECLIPSING BINARIES. PAPER 27: THE TOTALLY-ECLIPSING SYSTEM UZ DRACONIS

By John Southworth

Astrophysics Group, Keele University

UZ Dra is a detached and totally-eclipsing binary containing two late-F stars in a circular orbit of period 3·261 d. It has been observed by the *Transiting Exoplanet Survey Satellite* in 41 sectors, yielding a total of 664 809 high-quality flux measurements. We model these data and published radial velocities to determine the physical properties of the system to high precision. The masses of the stars are 1·291 ± 0·012 $M_{\odot}$ and 1·193 ± 0·009 $M_{\odot}$, and their radii are 1·278 ± 0·004 $R_{\odot}$ and 1·122 ± 0·003 $R_{\odot}$. The high precision of the radius measurements is made possible by the (previously unrecorded) total eclipses and the extraordinary amount of data available. The light-curves show spot modulation at the orbital period, and both stars rotate synchronously. Our determination of the distance to the system, 185·7 ± 2·4 pc, agrees very well with the parallax distance of 185·39 ± 0·39 pc from *Gaia* DR3. The properties of the system are consistent with theoretical predictions for an age of 600 ± 200 Myr and a slightly super-solar metallicity.